Real Convergence in the Mandelbrot Set

Chachayma Farfan, Diego — Research and Review
Corazzin, Teresa — Calculation and Review
Ge, Yang — Review
Schlicht, Robert — Draft Compilation

December 8, 2014

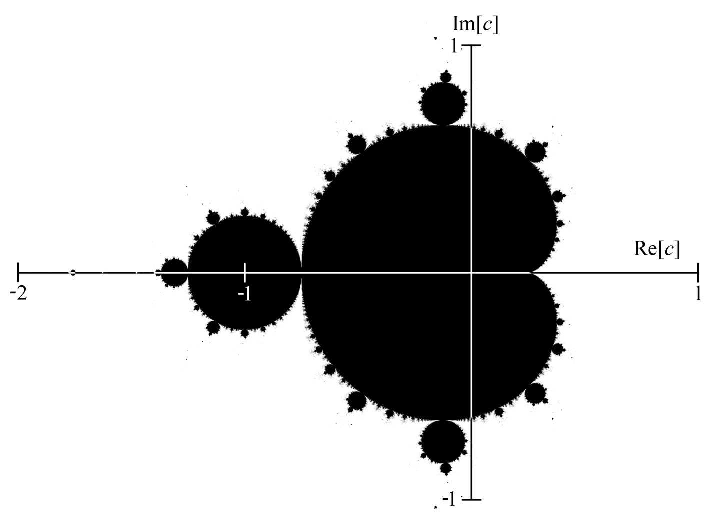
Abstract

We examine the convergence of real numbers in the Mandelbrot set, including what values belong to the set and how different values behave differently under the Mandelbrot mapping. We carry out this analysis with the aid of the logistic map and demonstrate a correspondence between the two mappings.

The Mandelbrot set M is defined as the set of points $c \in \mathbb{C}$ such that the recurrence relation

$$z_0 = 0$$
$$z_{n+1} = z_n^2 + c$$

remains bounded for all $n \in \mathbb{N}$. The sequence z_n need not converge to only a single value, or any specific value at all; as long as the sequence does not escape to infinity, it is included in M.
It is of interest to consider the intersection of M with $\mathbb{R}$; this is the set of real numbers contained in M. When c is real, the sequence z_n is real for every $n \in \mathbb{N}$.



The Mandelbrot set in the complex plane.

If for any point $c \in \mathbb{C}$, $|z_n| > 2$ for any $n \in \mathbb{N}$, z_n will escape to infinity.
We can see this from the triangle inequality, which states that $|z_{n+1}| = |z_n^2 + c| \geq |z_n^2| - |c|$. Suppose

$|c| = 2 + \epsilon$ for some $\epsilon > 0$ and for some $m \in \mathbb{N}$, $|z_m| > 2 + m\epsilon$; then

$$|z_{m+1}| \geq |z_m^2| - |c| > (2 + m\epsilon)^2 - (2 + \epsilon) > 2 + (m + 1)\,\epsilon.$$

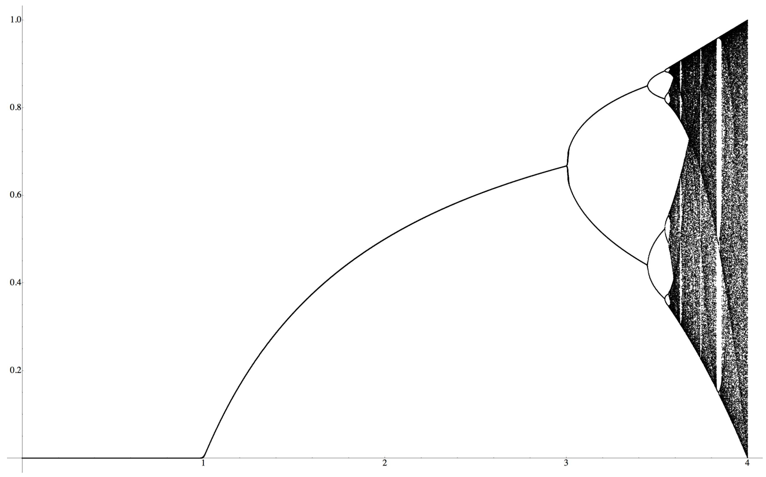
Since $|z_{n+1}| > 2 + (n + 1)\,\epsilon$ for any $n > m$, and the right hand side grows without bound as $n \to \infty$, the left hand side grows without bound.

The set M intersects with the real axis along the interval $[-2, 0.25]$. To show this, we inspect the interval from left to right.
By the previous result, any $c < -2$ cannot be in M. For the point $c = -2$, it is simple to show that $z_n = 2$ for every $n \geq 1$, so $-2 \in M$. If $-2 < c < 0$, $z_n < 2$ for every n, so c is also in M.
When $c > 0$, $z_{n+1} > z_n$ for every n; to remain bounded, this must approach a limit such that $z_{n+1} = z_n$. Expanding and solving for z_n yields $z_n = \frac{1+\sqrt{1-4c}}{2}$, which becomes non-real when $c > 0.25$. Thus 0.25 is the largest real number contained in M.

Having established that the interval $[-2, 0.25] \in M$, it is natural to ask if or how the sequences z_n converge for each point in the interval. To aid in this analysis, we will transform the equation using the substitutions $z_n = r(\frac{1}{2} - x_n)$ and $c = \frac{r}{2} - \frac{r^2}{4}$.

$$\begin{aligned}
z_{n+1} &= z_n^2 + c \\
z_{n+1} &= r^2\left(\frac{1}{2} - x_n\right)^2 + \left(\frac{r}{2} - \frac{r^2}{4}\right) \\
z_{n+1} &= r^2x_n^2 - r^2x_n + \frac{r^2}{4} + \frac{r}{2} - \frac{r^2}{4} \\
z_{n+1} &= r^2x_n^2 - r^2x_n + \frac{r}{2} \\
r\left(\frac{1}{2} - x_{n+1}\right) &= r^2x_n^2 - r^2x_n + \frac{r}{2} \\
\frac{r}{2} - rx_{n+1} &= r^2x_n^2 - r^2x_n + \frac{r}{2} \\
x_{n+1} &= rx_n - rx_n^2 \\
x_{n+1} &= rx_n(1 - x_n).
\end{aligned}$$

This recurrence relation is known as the logistic map. Because the relation for the logistic map can be derived from the Mandelbrot relation (and vice versa) using a linear change of variable, the convergence of the logistic map mirrors the convergence of $M \cap \mathbb{R}$.



Bifurcation diagram for the logistic map, $0 < r < 4$.

Solving for r in our second substitution yields $r = 1 + \sqrt{1-4c}$; as c ranges from -2 to 0.25, r ranges from 1 to 4. Therefore, we examine the behavior of the logistic map with r in the interval $[1, 4]$.

When $r \in [1, 3]$, x_n converges to a single nonzero point. From our substitution, this corresponds to the interval $c \in [-0.75, 0.25]$ in M. The point $c = -0.75$ forms the cusp between the main cardioid of M and the secondary cardioid to its left, and all complex c inside the main cardioid converge to a single point dependent on c.
We can find the point to which x_n converges for a given r by solving the equation

$$x = rx(1-x),$$

which simplifies to

$$rx^2 - (r-1)x = 0.$$

This equation represents the stable point mapping back to itself after one iteration. It has a degenerate solution at $x = 0$ and a second solution at $x = \frac{r-1}{r}$. We can then use this point our substitution to find the respective c and z values in M. For example, for $r = 1 + \sqrt{3}$, $x_n \to \frac{3-\sqrt{3}}{2}$ as $n \to \infty$, and for its analog $c = -0.5$, $z_n \to \frac{1-\sqrt{3}}{2}$ as $n \to \infty$.

As r increases to within the rage $(3, 1+\sqrt{6}]$, the logistic map becomes less stable; the map oscillates between two points as n becomes arbitrarily large. This interval corresponds to the interval $c \in [-1.25, -0.75)$ in M. To find the points to which x_n converges, we need to find two points that map back to themselves after exactly two iterations. We can find this by solving the equation

$$x = r(rx(1-x))(1-(rx(1-x))),$$

which simplifies to

$$r^3x^4 - 2r^3x^3 + (r^3 + r^2)x^2 - (r^2 + 1)x = 0.$$

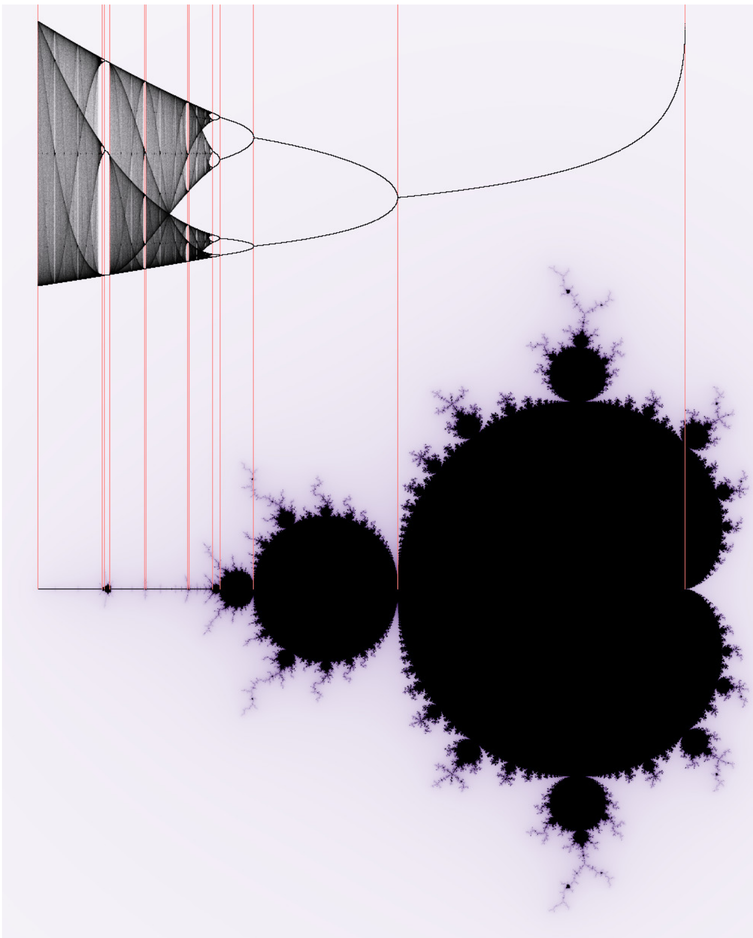
This equation again has a degenerate solution at $x = 0$ and three other solutions at

$$\begin{aligned} x &= \frac{r-1}{r} \\ &= \frac{r+1-\sqrt{r^2-2r-3}}{2r} \\ &= \frac{r+1+\sqrt{r^2-2r-3}}{2r}. \end{aligned}$$

The first of these three solutions maps to itself after every iteration and not after exactly two, so it is not valid. The other two solutions represent the two values between which x_n will oscillate as $n \to \infty$.
For example, for $r = 1 + \sqrt{5}$, x_n oscillates between $\frac{1}{2}$ and $\frac{1+\sqrt{5}}{4}$ as $n \to \infty$, and for its analog $c = -0.5$, z_n oscillates between -1 and 0.

When r is greater than $1 + \sqrt{6}$ and less than approximately 3.54409..., x_n will oscillate between 4 values as n increases. (The exact value for the upper bound of the range of 4-cycles is a root of a polynomial of degree 12). Within the corresponding range in M, where $c \in [-1.38084..., -0.75)$, z_n will oscillate between 4 values as well.
As r increases further, x_n oscillates between 8 values, then 16, and so on. The ratio of the lengths of each oscillation interval approaches 4.99620..., known as the Feigenbaum constant.

At approximately $r \geq 3.56995...$, the logistic map becomes "chaotic:" in the Mandelbrot set, any c off the real line does not belong in M. Exceptions occur at certain values, such as $r > 1 + 2\sqrt{2}$, after which x_n briefly oscillates between 3 values, then 6, 12, and so on.



Correspondence between the logistic map and the Mandelbrot set.

References

[1] Kurzweg, Ulrich. *Paths to Convergence for the Mandelbrot Set* [PDF document]. Retrieved from http://www2.mae.ufl.edu/ uhk/

[2] McMullen, Faben (2011). *Chaos on the Mandelbrot Needle*. Retrieved from https://sites.google.com/site/fabstellar84/fractals/real_chaos

[3] Murray, Richard (2006). *The Logistic Map* [PDF document]. Retrieved from https://www.cds.caltech.edu/ murray/wiki/index.php/Main_Page

[4] *The Mandelbrot Set and Julia Sets*. Retrieved from http://classes.yale.edu/fractals/Mandelset/MandelDef/LogisticMand/

[5] Weisstein, Eric (2014). *Logistic Map*. Retrieved from http://mathworld.wolfram.com/LogisticMap.html

[6] Yang, Jianke. *Julia Sets and the Mandelbrot Set* [PDF document]. Retrieved from http://www.emba.uvm.edu/ jxyang/teaching/